HEEGAARD SPLITTINGS OF DISTANCE EXACTLY n

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Abstract. In this paper, we show that, for any integer $n > 0$ and any integer $g > 1$, there exist genus-g Heegaard splittings of compact 3-manifolds with distance exactly n. Moreover, we show that for each $g(> 1)$, there exists a constant K_g such that, for each $n \geq K_g$, there exist genus-g Heegaard splittings of closed 3-manifolds with distance exactly n.

1. Introduction

For a closed orientable 3-manifold M, we say that $V_1 \cup_S V_2$ is a *Heegaard splitting* of M if V_1, V_2 are handlebodies such that $M = V_1 \cup V_2$ and $\partial V_1 = \partial V_2 = S$. Heegaard splittings of compact orientable 3-manifolds with nonempty boundaries can be defined similarly (see Section 2). Hempel [5] introduced the concept of *distance* of a Heegaard splitting by using the curve complex $C(S)$ of S, and showed that there exist arbitrarily high distance Heegaard splittings for closed 3-manifolds by using a construction of Kobayashi [6]. The manifolds are obtained by gluing two copies of handlebodies along their boundaries by the nth power of a pseudo-Anosov map for each n. Abrams and Schleimer [1] showed that the distance of the Heegaard splitting grows linearly with respect to n by using the result of Masur and Minsky [8]. Moreover, Evans [3] gave a combinatorial method to construct Heegaard splittings of high distance. The main purpose of this paper is to give an answer to the following question.

Question Given $g > 1$ and $n \geq 0$, does there exist a genus-g Heegaard splitting with distance exactly n?

For certain values, there are known examples that answer the above question affirmatively. For example, Berge and Scharlemann [2] showed that there exist 3-manifolds each of which admits genus-2 Heegaard splittings with distance exactly 3.

In this paper, we first construct Heegaard splittings of compact orientable 3-manifolds with nonempty boundaries which have distance exactly n.

Theorem 1.1. *For any integer $n > 0$ and any integer $g > 1$, there exists a genus-g Heegaard splitting $C_1 \cup_P C_2$ with distance exactly n, where C_1 and C_2 are compression bodies.*

To prove Theorem 1.1, we give a method of constructing a pair of curves with distance exactly n. In fact, Schleimer [12] gave a method of constructing a pair of curves with distance exactly four on the five-holed sphere by using *subsurface projection maps* defined by Masur and Minsky [9] (for the definition, see Section 2). In Section 4, we mimic the idea of Schleimer to construct a pair of curves with distance exactly n for any positive integer n. By using the pair of curves and the

properties of a compression body obtained by adding a 1-handle to $S \times [0,1]$ where S is a closed surface (for detail, see Section 3), we give the proof of Theorem 1.1.

Moreover, once we have a Heegaard splitting of a compact orientable 3-manifold with distance exactly n, then we can construct from it a Heegaard splitting of a closed orientable 3-manifold with the same distance by using the result by Ma and Qiu [7, Theorems 1.3 and 5.3]. Hence, we obtain:

Corollary 1.2. *For each $g(>1)$, there exists a constant K_g such that, for each $n \geq K_g$, there exist genus-g Heegaard splittings of closed 3-manifolds with distance exactly n.*

2. Preliminaries

Let S be a compact orientable surface with genus g and p boundary components. A simple closed curve in S is *essential* if it does not bound a disk in S and is not parallel to ∂S. An arc properly embedded in S is *essential* if it does not co-bound a disk in S together with an arc on ∂S. We say that S is *sporadic* if $g = 0, p \leq 4$ or $g = 1, p \leq 1$. We say that S is *simple*, if S contains no essential simple closed curves. We note that S is simple if and only if S is a 2-sphere with at most three boundary components. A subsurface X in S is *essential* if each component of ∂X is contained in ∂S or is essential in S.

Heegaard splittings

A 3-manifold V is a *compression body* if there exists a closed (possibly empty) surface F and a 0-handle B such that V is obtained from $F \times [0,1] \cup B$ by adding 1-handles to $F \times \{1\} \cup \partial B$. The subsurface of ∂V corresponding to $F \times \{0\}$ is denoted by $\partial_- V$. Then $\partial_+ V$ denotes the subsurface $\partial V \setminus \partial_- V$ of ∂V. A compression body V is called a *handlebody* if $\partial_- V = \emptyset$. A disk D properly embedded in V is called an *essential disk* if ∂D is an essential simple closed curve in $\partial_+ V$.

Let M be a compact orientable 3-manifold. We say that $C_1 \cup_P C_2$ is a *genus-g Heegaard splitting* of M if C_1 and C_2 are compression bodies in M such that $C_1 \cup C_2 = M$ and $C_1 \cap C_2 = \partial_+ C_1 = \partial_+ C_2 = P$.

Curve complexes

Except in sporadic cases, the *curve complex* $\mathcal{C}(S)$ is defined as follows: each vertex of $\mathcal{C}(S)$ is the isotopy class of an essential simple closed curve on S, and a collection of $k+1$ vertices forms a k-simplex of $\mathcal{C}(S)$ if they can be realized by disjoint curves in S. In sporadic cases, we need to modify the definition of the curve complex slightly, as follows. Note that the surface S is simple unless S is a torus, a torus with one boundary component, or a sphere with 4 boundary components. When S is a torus or a torus with one boundary component (resp. a sphere with 4 boundary components), a collection of $k+1$ vertices forms a k-simplex of $\mathcal{C}(S)$ if they can be realized by curves in S which mutually intersect exactly once (resp. twice). The *arc-and-curve complex* $\mathcal{AC}(S)$ is defined similarly, as follows: each vertex of $\mathcal{AC}(S)$ is the isotopy class of an essential properly embedded arc or an essential simple closed curve on S, and a collection of $k+1$ vertices forms a k-simplex of $\mathcal{AC}(S)$ if they can be realized by disjoint arcs or simple closed curves in S. Throughout this paper, for a vertex $x \in \mathcal{C}(S)$ we often abuse notations and use x to represent (the isotopy class of) a geometric representative of x.

For two vertices x, y of $\mathcal{C}(S)$, we define the *distance* $d_{\mathcal{C}(S)}(x, y)$ between x and y, which will be denoted by $d_S(x, y)$ in brief, as the minimal number of 1-simplexes

of a simplicial path in $\mathcal{C}(S)$ joining x and y. Let X, Y be subsets of the vertices of $\mathcal{C}(S)$. Then we define $\mathrm{diam}_S(X,Y) = \mathrm{diam}_S(X \cup Y)$. Similarly, we can define the distance $d_{\mathcal{AC}(S)}(x,y)$ and $\mathrm{diam}_{\mathcal{AC}(S)}(X,Y)$.

For a sequence $a_0, a_1, \ldots, a_n$ of vertices in $\mathcal{C}(S)$ with $a_i \cap a_{i+1} = \emptyset$ $(i = 0, 1, \ldots, n-1)$, we denote by $[a_0, a_1 \ldots, a_n]$ the path in $\mathcal{C}(S)$ with vertices $a_0, a_1 \ldots, a_n$ in this order. We say that a path $[a_0, a_1 \ldots, a_n]$ is a *geodesic* if $n = d_s(a_0, a_n)$.

Let V be a compression body. Then the *disk complex* $\mathcal{D}(V)$ is the subcomplex of $\mathcal{C}(\partial_+ V)$ consisting of the vertices with representatives bounding disks of V. For a genus-$g(\geq 2)$ Heegaard splitting $C_1 \cup_P C_2$, the (Hempel) *distance* of $C_1 \cup_P C_2$ is defined by $d_P(\mathcal{D}(C_1), \mathcal{D}(C_2)) = \min\{d_P(x,y) \mid x \in \mathcal{D}(C_1), y \in \mathcal{D}(C_2)\}$.

Subsurface projection maps

Let $\mathcal{P}(Y)$ denote the power set of a set Y. Suppose that X is an essential non-simple subsurface of S. We call the composition $\pi_0 \circ \pi_A$ of maps $\pi_A : \mathcal{C}^0(S) \to \mathcal{P}(\mathcal{AC}^0(X))$ and $\pi_0 : \mathcal{P}(\mathcal{AC}^0(X)) \to \mathcal{P}(\mathcal{C}^0(X))$ a *subsurface projection* if they satisfy the following: for a vertex α, take a representative α so that $|\alpha \cap X|$ is minimal, where $|\cdot|$ is the number of connected components. Then

- $\pi_A(\alpha)$ is the set of all isotopy classes of the components of $\alpha \cap X$,
- $\pi_0(\{\alpha_1, \ldots, \alpha_n\})$ is the union for all $i = 1, \ldots, n$ of the set of all isotopy classes of the components of $\partial N(\alpha_i \cup \partial X)$ which are essential in X, where $N(\alpha_i \cup \partial X)$ is a regular neighborhood of $\alpha_i \cup \partial X$ in X.

We say that α *misses* X (resp. α *cuts* X) if $\alpha \cap X = \emptyset$ (resp. $\alpha \cap X \neq \emptyset$).

Lemma 2.1. *Let X be as above. Let $[\alpha_0, \alpha_1, ..., \alpha_n]$ be a path in $\mathcal{C}(S)$ such that every α_i cuts X. Then* $\mathrm{diam}_X(\pi_X(\alpha_0), \pi_X(\alpha_n)) \leq 2n$.

Proof. Since $d_S(\alpha_i, \alpha_{i+1}) = 1$ and every α_i cuts X, we have

$$\mathrm{diam}_{\mathcal{AC}(X)}(\pi_A(\alpha_i), \pi_A(\alpha_{i+1})) \leq 1$$

for every $i = 0, 1, \ldots, n-1$. This together with [9, Lemma 2.2] implies

$$\mathrm{diam}_X(\pi_0(\pi_A(\alpha_i)), \pi_0(\pi_A(\alpha_{i+1})))(= \mathrm{diam}_X(\pi_X(\alpha_i), \pi_X(\alpha_{i+1}))) \leq 2.$$

Since $\mathrm{diam}_X(\pi_X(\alpha_0), \pi_X(\alpha_n)) \leq \sum_{i=0}^{n-1} \mathrm{diam}_X(\pi_X(\alpha_i), \pi_X(\alpha_{i+1}))$, this implies

$$\mathrm{diam}_X(\pi_X(\alpha_0), \pi_X(\alpha_n)) \leq 2n.$$

□

Remark 2.2. If X is an essential subsurface of S with at least two components, then for any pair of curves α, α' on S we have $\mathrm{diam}_X(\pi_X(\alpha), \pi_X(\alpha')) \leq 2$. To be precise, let X_1 be one of the components of X, and X_2 the union of the others. Let a and a' be elements of $\pi_X(\alpha)$ and $\pi_X(\alpha')$, respectively. If both a and a' are contained in X_i for some $i = 1, 2$, say X_1, then we can find a curve on X_2 that is disjoint from $a \cup a'$, which implies $d_X(a, a') \leq 2$. If $a \subset X_1$ and $a' \subset X_2$ (or $a \subset X_2$ and $a' \subset X_1$), we have $d_X(a, a') \leq 1$. Thus $d_X(a, a') \leq 2$ for any pair of elements $a \in \pi_X(\alpha)$ and $a' \in \pi_X(\alpha')$, and hence we have $\mathrm{diam}_X(\pi_X(\alpha), \pi_X(\alpha')) \leq 2$.

3. Disk complexes

Let $\mathcal{D}(V)\,(\subset \mathcal{C}(\partial_+ V))$ be the disk complex of a compression body V. We have a decomposition $\mathcal{D}(V) = \mathcal{D}_{\mathrm{nonsep}}(V) \sqcup \mathcal{D}_{\mathrm{sep}}(V)$, where $\mathcal{D}_{\mathrm{nonsep}}(V)$ (resp. $\mathcal{D}_{\mathrm{sep}}(V)$) denotes the subset of $\mathcal{D}(V)$ consisting of the vertices with representatives bounding non-separating (resp. separating) essential disks of V. In this section, we prove the following proposition.

Proposition 3.1. *Let V be a compression body obtained by adding a 1-handle to $F \times [0, 1]$, where F is a genus-$(g-1)$ closed orientable surface ($g > 1$). Then we have the following.*

1. *$\mathcal{D}_{\mathrm{nonsep}}(V)$ consists of a point, say c_0.*
2. *For each element c_α of $\mathcal{D}_{\mathrm{sep}}(V)$, there is a 1-simplex in $\mathcal{C}(\partial_+ V)$ joining c_0 and c_α.*

Remark 3.2. In fact, we can see that $\mathcal{D}_{\mathrm{sep}}(V)$ is a countable, infinite set and that there is no 1-simplex between c_α and $c_{\alpha'}$, for each pair $c_\alpha, c_{\alpha'} \in \mathcal{D}_{\mathrm{sep}}(V)$.

In the remaining of this section, V denotes a compression body obtained by adding a 1-handle to $F \times [0, 1]$, where F is a genus-$(g-1)$ closed orientable surface ($g > 1$). Then D denotes the essential disk of V corresponding to the co-cre of the 1-handle. Proposition 3.1 follows from Lemmas 3.3 and 3.4 below.

Lemma 3.3. *Any non-separating disk properly embedded in V is ambient isotopic to D.*

Proof. Let D' be a non-separating disk in V. Assume that D and D' intersect transversely, and $|D \cap D'|$ is minimized up to ambient isotopy class of D'.

Suppose $|D \cap D'| = 0$, i.e., $D \cap D' = \emptyset$. Then D' is properly embedded disk in the manifold obtained from V by cutting along D, that is, $F \times [0, 1]$. Since any disk properly embedded in $F \times [0, 1]$ is boundary parallel and D' is non-separating in V, we see that $D \cup D'$ bounds a product region, and hence D' is ambient isotopic to D.

Suppose $|D \cap D'| > 0$. By standard innermost disk arguments, we can see that $D \cap D'$ has no loop components. Note that there are at least two components of $D' \setminus (D \cap D')$ which are outermost in D'. Take a pair of such outermost components, say Δ_1 and Δ_2, which are the next to each other, i.e., there is a subarc $\beta \subset \partial D'$ such that $\beta \cap \Delta_1$ is an endpoint of β and $\beta \cap \Delta_2$ is the other endpoint of β, and β dost not intersect any other outermost disk of $D' \setminus (D \cap D')$. Note that we can retrieve $F \times [0, 1]$ by cutting V along D. Let D^+, D^- be the copies of D in $F \times \{1\}$, and let $\overline{\Delta}_1$ (resp. $\overline{\Delta}_2$) be the closure of the image of Δ_1 (resp. Δ_2) in $F \times [0, 1]$. Note that $\overline{\Delta}_1$ and $\overline{\Delta}_2$ are disks properly embedded in $F \times [0, 1]$, and $\overline{\Delta}_i \cap (D^+ \cup D^-)$ consists of an arc properly embedded in $D^+ \cup D^-$. Let Γ_i ($i = 1, 2$) be the disk in $F \times \{1\}$ such that $\partial \Gamma_i = \partial \overline{\Delta}_i$. Without loss of generality, we may suppose $\overline{\Delta}_1 \cap (D^+ \cup D^-) = \overline{\Delta}_1 \cap D^+$. Note that if D^- is not contained in Γ_1, we can isotope D' in V via the product region between $\overline{\Delta}_1$ and Γ_1 to reduce $|D \cap D'|$, a contradiction. Hence, D^- is contained in Γ_1. Let β be the arc in $\partial D'$ as above. Then $\beta \cap D$ consists of finite number of points, say $p_0, p_1, \ldots, p_n$, where $\partial \beta = \{p_0, p_n\}$, $p_0 \in \partial \overline{\Delta}_1$, $p_n \in \partial \overline{\Delta}_2$, and $p_0, p_1, \ldots, p_n$ are arrayed on β in this order. Then a small neighborhood of p_0 in β is contained in a small neighborhood of D^- in $F \times [0, 1]$. If the other endpoint of the subarc $\overline{p_0 p_1}$ of β is contained in ∂D^-, then we see

that the subarc $\overline{p_0p_1}$ is an inessential arc in $\mathrm{Cl}(F\times\{1\}\setminus(D^+\cup D^-))$. This shows that we can reduce $|D\cap D'|$ by an isotopy on D', a contradiction. By applying the same argument successively, we see that each subarc $\overline{p_{i-1}p_i}$ $(i=1,2,\ldots,n)$ joins D^+ and D^-, and particularly, a small neighborhood of p_n in β is contained in a small neighborhood of D^+. This shows that $\overline{\Delta}_2\cap(D^+\cup D^-)=\overline{\Delta}_2\cap D^-$. Then we see that D^+ is not contained in Γ_2, hence we have a contradiction by using the argument as above. □

Let D' be a separating essential disk properly embedded in V. By an argument similar to that in the proof of Lemma 3.3, we can see that D' is ambient isotopic to a disk disjoint from D. Hence, we have the following lemma.

Lemma 3.4. *Any separating essential disk properly embedded in V can be isotoped to be disjoint from the non-separating disk D.*

4. A pair of curves with distance exactly n

In this section, we construct pairs of curves with distance exactly n. Let S be a closed orientable surface with genus greater than 1. We first prove the following two propositions. Then we describe the constructions of sequences of curves of length n and show that they give geodesics in $\mathcal{C}(S)$.

Proposition 4.1. *For an even positive integer $n(\geq 4)$, let $[\alpha_0,\alpha_1,\ldots,\alpha_n]$ be a path in $\mathcal{C}(S)$ satisfying the following.*

(H1) *$[\alpha_0,\ldots,\alpha_{n-2}]$ and $[\alpha_{n-2},\alpha_{n-1},\alpha_n]$ are geodesics in $\mathcal{C}(S)$,*

(H2) *$\mathrm{diam}_{X_{n-2}}(\pi_{X_{n-2}}(\alpha_{n-4}),\pi_{X_{n-2}}(\alpha_n))\geq 4n$, where $X_{n-2}=\mathrm{Cl}(S\setminus N(\alpha_{n-2}))$.*

Then $[\alpha_0,\alpha_1,\ldots,\alpha_n]$ is a geodesic in $\mathcal{C}(S)$.

Remark 4.2. In Proposition 4.1, we note that X_{n-2} is connected. This can be shown by using Remark 2.2 together with the condition (H2).

Proof of Proposition 4.1. Let $[\beta_0,\beta_1,\ldots,\beta_m]$ be a geodesic in $\mathcal{C}(S)$ such that $\beta_0=\alpha_0$, $\beta_m=\alpha_n$. Then $m\leq n$.

Claim 4.3. *$\beta_j=\alpha_{n-2}$ for some $j\in\{0,1,\ldots,m\}$.*

Proof. Assume on the contrary that $\beta_j\neq\alpha_{n-2}$ for any j. Then every β_j cuts X_{n-2}. By Lemma 2.1, we have $\mathrm{diam}_{X_{n-2}}(\pi_{X_{n-2}}(\beta_0),\pi_{X_{n-2}}(\beta_m))\leq 2m$. Since $[\alpha_0,\alpha_1,\ldots,\alpha_{n-2}]$ is a geodesic, no α_i $(0\leq i\leq n-3)$ is isotopic to α_{n-2}. Hence each α_i $(0\leq i\leq n-3)$ cuts X_{n-2}. By Lemma 2.1, $\mathrm{diam}_{X_{n-2}}(\pi_{X_{n-2}}(\alpha_0),\pi_{X_{n-2}}(\alpha_{n-4}))\leq 2(n-4)<2n$. Hence,

$$\begin{aligned}\mathrm{diam}_{X_{n-2}}(\pi_{X_{n-2}}(\alpha_{n-4}),\pi_{X_{n-2}}(\alpha_n)) &\leq \mathrm{diam}_{X_{n-2}}(\pi_{X_{n-2}}(\alpha_{n-4}),\pi_{X_{n-2}}(\alpha_0))\\ &\quad+\mathrm{diam}_{X_{n-2}}(\pi_{X_{n-2}}(\alpha_0),\pi_{X_{n-2}}(\alpha_n))\\ &< 2n+\mathrm{diam}_{X_{n-2}}(\pi_{X_{n-2}}(\beta_0),\pi_{X_{n-2}}(\beta_m))\\ &\leq 2n+2m\\ &\leq 4n.\end{aligned}$$

This contradicts the hypothesis (H2). □

By Claim 4.3 and the hypothesis (H1), we have the equalities

$$\begin{aligned} j=d_S(\beta_0,\beta_j) &= d_S(\alpha_0,\alpha_{n-2})=n-2,\\ m-j=d_S(\beta_j,\beta_m) &= d_S(\alpha_{n-2},\alpha_n)=2.\end{aligned}$$

By combining the above equalities, we have $m = n$. Recall that $[\beta_0, \beta_1, \ldots, \beta_m]$ is a geodesic in $\mathcal{C}(S)$ with $\beta_0 = \alpha_0$ and $\beta_m = \alpha_n$. Hence, $[\alpha_0, \alpha_1, \ldots, \alpha_n]$ is a geodesic in $\mathcal{C}(S)$. □

Proposition 4.4. *For a positive integer n, let $[\alpha_0, \alpha_1, \ldots, \alpha_n]$ be a path in $\mathcal{C}(S)$ satisfying the following.*

- (H1') $[\alpha_0, \ldots, \alpha_{n-1}]$ *and* $[\alpha_{n-2}, \alpha_{n-1}, \alpha_n]$ *are geodesics in* $\mathcal{C}(S)$*,*
- (H2') $\mathrm{diam}_{S'}(\pi_{S'}(\alpha_0), \pi_{S'}(\alpha_n)) > 2n$*, where* $S' = \mathrm{Cl}(S \setminus N(\alpha_{n-2} \cup \alpha_{n-1}))$*.*

Then $[\alpha_0, \alpha_1, \ldots, \alpha_n]$ is a geodesic in $\mathcal{C}(S)$.

Remark 4.5. In Proposition 4.4, we note that S' is connected. This can be shown by using Remark 2.2 together with the condition (H2').

Proof of Proposition 4.4. Let $[\beta_0, \beta_1, \ldots, \beta_m]$ be a geodesic in $\mathcal{C}(S)$ such that $\beta_0 = \alpha_0$, $\beta_m = \alpha_n$. Then $m \leq n$.

Claim 4.6. *There exists $j \in \{0, 1, \ldots, m\}$ such that $\beta_j = \alpha_{n-2}$ or $\beta_j = \alpha_{n-1}$.*

Proof. Suppose that $\beta_j \neq \alpha_{n-2}$ and $\beta_j \neq \alpha_{n-1}$ for any j. Then each β_j cuts S'. Hence, by Lemma 2.1, we have

$$\mathrm{diam}_{S'}(\pi_{S'}(\alpha_0), \pi_{S'}(\alpha_n)) = \mathrm{diam}_{S'}(\pi_{S'}(\beta_0), \pi_{S'}(\beta_m)) \leq 2m.$$

On the other hand, by (H2'), $\mathrm{diam}_{S'}(\pi_{S'}(\beta_0), \pi_{S'}(\beta_m)) > 2n$, a contradiction. □

Suppose $\beta_j = \alpha_{n-2}$. Then we have the equalities

$$\begin{aligned} j = d_S(\beta_0, \beta_j) &= d_S(\alpha_0, \alpha_{n-2}) = n - 2, \\ m - j = d_S(\beta_j, \beta_m) &= d_S(\alpha_{n-2}, \alpha_n) = 2. \end{aligned}$$

By combining the above equalities, we have $n = m$. Hence, $[\alpha_0, \alpha_1, \ldots, \alpha_n]$ is a geodesic in $\mathcal{C}(S)$.

We can use a similar argument for the case when $\beta_j = \alpha_{n-1}$. This completes the proof of Proposition 4.4. □

For a given positive integer n, we construct a geodesic $[\alpha_0, \alpha_1, \ldots, \alpha_n]$ in $\mathcal{C}(S)$, i.e., $d_S(\alpha_0, \alpha_n) = n$, by using Propositions 4.1 and 4.4.

4.1. **A construction of a concrete example: the case when n is even.** We first assume that n is even. Let α_0, α_2 be essential non-separating simple closed curves on S which intersect transversely in one point, and let α_1 be an essential simple closed curve on S which is disjoint from $\alpha_0 \cup \alpha_2$. Let $X_2 = \mathrm{Cl}(S \setminus N(\alpha_2))$. Note that $[\alpha_0, \alpha_1, \alpha_2]$ is a geodesic of length two in $\mathcal{C}(S)$. Choose a homeomorphism $f_2 : S \to S$ such that $f_2(N(\alpha_2)) = N(\alpha_2)$ and that $\mathrm{diam}_{X_2}(\pi_{X_2}(\alpha_0), \pi_{X_2}(f_2(\alpha_0))) \geq 4n$. This is possible by [8, Proposition 4.6]. Let $\alpha_3 = f_2(\alpha_1)$ and $\alpha_4 = f_2(\alpha_0)$. Note that $[\alpha_2, \alpha_3, \alpha_4]$ is a geodesic of length two in $\mathcal{C}(S)$.

We repeat this process to construct a path $[a_0, a_1, \ldots, a_n]$. Namely, for each even $i \in \{2, 4, \ldots, n-2\}$,

- (i-1) $X_i = \mathrm{Cl}(S \setminus N(\alpha_i))$,
- (i-2) $f_i : S \to S$ is a homeomorphism such that $f_i(N(\alpha_i)) = N(\alpha_i)$ and that $\mathrm{diam}_{X_i}(\pi_{X_i}(\alpha_{i-2}), \pi_{X_i}(f_i(\alpha_{i-2}))) \geq 4n$,
- (i-3) $\alpha_{i+1} = f_i(\alpha_{i-1})$ and $\alpha_{i+2} = f_i(\alpha_{i-2})$.

Note that $[\alpha_i, \alpha_{i+1}, \alpha_{i+2}]$ is a geodesic of length two in $\mathcal{C}(S)$.

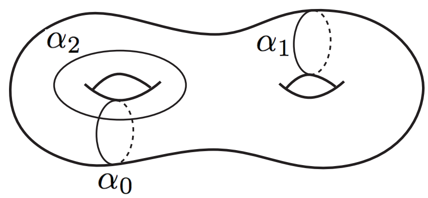

FIGURE 1.

Assertion 4.7. *For each $k \in \{2, 4, \dots, n\}$, a path $[\alpha_0, \alpha_1, \dots, \alpha_k]$ in $\mathcal{C}(S)$ is a geodesic.*

Proof. We prove the proposition by mathematical induction on k. It is clear that $[\alpha_0, \alpha_1, \alpha_2]$ is a geodesic in $\mathcal{C}(S)$. Hence, Assertion 4.7 holds for $k = 2$. Assume that $[\alpha_0, \alpha_1, \dots, \alpha_k]$ is a geodesic in $\mathcal{C}(S)$ for some $k \in \{2, 4, \dots, n-2\}$. We note that $[\alpha_k, \alpha_{k+1}, \alpha_{k+2}]$ is a geodesic in $\mathcal{C}(S)$. Furthermore, by the condition (i-2), we have $\mathrm{diam}_{X_k}(\pi_{X_k}(\alpha_{k-2}), \pi_{X_k}(\alpha_{k+2})) \geq 4n > 4k$. Hence, by Proposition 4.1, the path $[\alpha_0, \alpha_1, \dots, \alpha_{k+2}]$ is a geodesic in $\mathcal{C}(S)$, which shows that Assertion 4.7 holds for $k+2$. This completes the proof of Assertion 4.7. □

4.2. **A construction of a concrete example: the case when n is odd.** Suppose that n is odd. Let $[\alpha_0, \alpha_1, \dots, \alpha_{n-1}]$ be a path in $\mathcal{C}(S)$ as in the previous subsection. Here, in addition, we assume that each α_i is a non-separating curve. (This is done if α_1 is a non-separating curve.) Note that α_{n-3} intersects α_{n-1} transversely in one point and is disjoint from α_{n-2}. Note also that α_{n-2} is non-separating. It is easy to see that these imply that $\alpha_{n-1} \cup \alpha_{n-2}$ is non-separating. Choose a non-separating essential simple closed curve γ on S such that $\gamma \cap \alpha_{n-1} = \emptyset$ and γ intersects α_{n-2} transversely in one point. Let $S' = \mathrm{Cl}(S \setminus N(\alpha_{n-2} \cup \alpha_{n-1}))$. By [8, Proposition 4.6], there exists a homeomorphism $f : S \to S$ such that $f(S') = S'$ and $\mathrm{diam}_{S'}(\pi_{S'}(f(\gamma)), \pi_{S'}(\alpha_0)) > 2n$. Let $\alpha_n = f(\gamma)$. Note that $\alpha_n \cap \alpha_{n-1} = \emptyset$ and α_n intersects α_{n-2} transversely in one point. This fact implies that $[\alpha_{n-2}, \alpha_{n-1}, \alpha_n]$ is a geodesic in $\mathcal{C}(S)$. On the other hand, $[\alpha_0, \dots, \alpha_{n-1}]$ is also a geodesic in $\mathcal{C}(S)$. Hence, by Proposition 4.4 together with the above inequality $\mathrm{diam}_{S'}(\pi_{S'}(f(\gamma)), \pi_{S'}(\alpha_0)) > 2n$, we see that the path $[\alpha_0, \alpha_1, \dots, \alpha_n]$ is a geodesic in $\mathcal{C}(S)$.

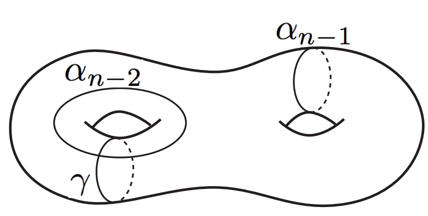

FIGURE 2.

We remark that the construction of geodesics introduced in this section works for compact surfaces with genus greater than 1.

5. Proofs of Theorem 1.1 and Corollary 1.2

Proof of Theorem 1.1. Let C_1 and C_2 be copies of the compression body obtained by adding a 1-handle to $F \times [0,1]$, where F is a genus-$(g-1)$ closed orientable surface ($g > 1$). Let α_0 be the boundary of the non-separating essential disk D_1 properly embedded in C_1 and α_2 a simple closed curve on $\partial_+ C_1$ which intersects α_0 transversely in one point. Then we construct a geodesic $[\alpha_0, \alpha_1, \ldots, \alpha_{n+2}]$ on $\partial_+ C_1$ as in Section 4. Note that α_{n+2} intersects α_n transversely in one point by the construction. Take any homeomorphism $h : \partial_+ C_2 \to \partial_+ C_1$ such that $h(\partial D_2) = \alpha_{n+2}$, where D_2 is the non-separating essential disk properly embedded in C_2. We identify the boundary components $\partial_+ C_1$ and $\partial_+ C_2$ by h, and let $P = \partial_+ C_1 = h(\partial_+ C_2)$. Then $C_1 \cup_P C_2$ is a genus-g Heegaard splitting of a compact orientable 3-manifold.

Let D_1' be a separating essential disk in C_1 disjoint from α_2 obtained as follows. First we take two disks D_1^+ and D_1^- parallel to D_1. Take the subarc of α_2 lying outside of the product region betwteen D_1^+ and D_1^-. Then D_1' is obtained from $D_1^+ \cup D_1^-$ by adding a band along the subarc of α_2. Similarly, we can obtain a separating essential disk D_2' in C_2 disjoint from α_n, by using D_2 and α_n. On the other hand, we have $d_P(\alpha_2, \alpha_n) = n - 2$ since $[\alpha_0, \alpha_1, \ldots, \alpha_{n+2}]$ is a geodesic in $\mathcal{C}(S)$. Hence,

$$\begin{aligned} d_P(\partial D_1', \partial D_2') &\leq d_P(\partial D_1', \alpha_2) + d_P(\alpha_2, \alpha_n) + d_P(\alpha_n, \partial D_2') \\ &= 1 + (n-2) + 1 \\ &= n. \end{aligned}$$

Let $D_1'' \subset C_1$ and $D_2'' \subset C_2$ be any essential disks. By Proposition 3.1, we have $d_P(\partial D_i'', \partial D_i) \leq 1$ for $i = 1, 2$. This implies

$$\begin{aligned} d_P(\partial D_1, \partial D_2) &\leq d_P(\partial D_1, \partial D_1'') + d_P(\partial D_1'', \partial D_2'') + d_P(\partial D_2'', \partial D_2) \\ &\leq 1 + d_P(\partial D_1'', \partial D_2'') + 1, \end{aligned}$$

and hence

$$\begin{aligned} d_P(\partial D_1'', \partial D_2'') &\geq d_P(\partial D_1, \partial D_2) - 2 \\ &= d_P(\alpha_0, \alpha_{n+2}) - 2 \\ &= (n+2) - 2 \\ &= n. \end{aligned}$$

Hence $d_P(\partial D_1'', \partial D_2'') \geq n$ for any pair of essential disks $D_1'' \subset C_1$ and $D_2'' \subset C_2$, which implies $d_P(\mathcal{D}(C_1), \mathcal{D}(C_2)) \geq n$. Since $d_P(\partial D_1', \partial D_2') \leq n$, we have $d_P(\mathcal{D}(C_1), \mathcal{D}(C_2)) = n$. □

Proof of Corollary 1.2. Let $C_1 \cup_P C_2$ be a genus-g Heegaard splitting with distance exactly n as above. Let $K_g := K + 2$, where K is the constant as in [10, Theorem 1.1]. Assume that $n \geq K_g$. By applying [7, Theorem 5.3], we obtain a genus-g Heegaard splitting $C_1 \cup_P C_2^*$ with distance exactly n, where C_2^* is a handlebody. Then by applying [7, Theorem 1.3] to $C_1 \cup_P C_2^*$, we obtain a genus-g Heegaard splitting $C_1^* \cup_P C_2^*$ with distance exactly n, where C_1^* and C_2^* are handlebodies. □

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Appendix A.

We apply the idea for the proof of Theorem 1.1 to genus-1 1-bridge knots. A knot K in an orientable closed 3-manifold M is called a *genus-1 1-bridge knot*, or a *(1,1)-knot* in brief, if $(M,K) = (V_1,t_1) \cup_P (V_2,t_2)$, where $V_1 \cup_P V_2$ is a genus-1 Heegaard splitting and t_i is a trivial arc in V_i $(i = 1,2)$. Then $(V_1,t_1) \cup_P (V_2,t_2)$ is called a *(1,1)-splitting* of (M,K). Let $P^* := P \setminus \partial t_i$. Then $\mathcal{D}(V_i)$ denotes the subset of $\mathcal{C}(P^*)$ consisting of the vertices with representatives bounding disks in $V_i \setminus t_i$. Then the *distance* of $(V_1,t_1) \cup_P (V_2,t_2)$ is defined by $d_{P^*}(\mathcal{D}(V_1),\mathcal{D}(V_2))$. (For details, see [11].)

Theorem A.1. *For any even integer $n > 0$, there exists a (1,1)-splitting with distance exactly n.*

Proof. Basically the proof of Theorem A.1 is the same as that of Theorem 1.1 for the case when n is even, so here we give just an outline of the arguments. The key fact is the following assertion proved by Saito [11, Proposition 3.8] which shows that $\mathcal{D}(V_i)$ has the same structure as $\mathcal{D}(C_i)$ in the proof of Theorem 1.1 $(i = 1,2)$.

Assertion A.2. *Let D_i be an essential disk in $V_i \setminus t_i$ as in Figure 3. Then any non-separating essential disk in $V_i \setminus t_i$ is isotopic to D_i and any separating essential disk in $V_i \setminus t_i$ can be isotoped to be disjoint from D_i.*

Then starting with simple closed curves $\alpha_0 = \partial D_1, \alpha_1$ and α_2 as in Figure 4, we apply the construction of a geodesic in Subsection 4.1. Then the arguments in the proof of Theorem 1.1 enable us to show the existence of a (1,1)-splitting with distance n for each even n. $\square$

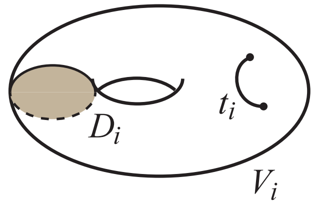

FIGURE 3.

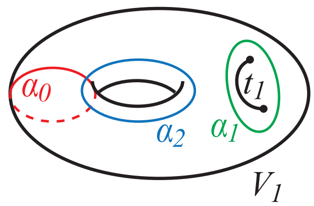

FIGURE 4.

Here we note that α_1 is separating in $\partial V_1 \setminus t_1$. Hence, we cannot apply the extension of the geodesic described in Subsection 4.2.

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